UNIQUENESS CONDITIONS AND METHOD OF SOLUTION

OF THE COEFFICIENT INVERSE PROBLEM OF THERMAL CONDUCTIVITY

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The uniqueness conditions are formulated for the coefficient problem of thermal conductivity and an approach to devising the algorithm for solving it is presented.

Consider the inverse thermal conductivity problem (ITCP), which consists in determining one of the coefficients of the thermal conductivity equation with respect to the known temperature $T(x, \tau)$ at the instant of time $\tau = t = \text{const} > 0$, and the assigned values of the temperature and the thermal flux density at the region boundary at any instant of time.

The uniqueness of solution of such ITCPs for linear and nonlinear equations was investigated earlier in [1-12]. For the nonlinear case, uniqueness theorems "in the small" were obtained in [5-6], while the uniqueness theorems in [7] were obtained under the assumption that the sought coefficient belongs to a certain contiguity class. We shall derive below the uniqueness theorem "in the large." It is required basically that the sought coefficient belong to class C (more detailed presentation is given below). This theorem is proved by means of roughly the same method as the one used for proving Theorem 4 in [10] (see also remarks at the end of [10, 11]). In contrast to [10], where the problem is considered in the half-space (see also [8]), we shall consider the end section, which involves additional difficulties resulting from the consideration of conditions at its other end. The temperature and density of the thermal flux must be assigned at both ends of the section.

The second part of this article provides an approach to the numerical realization of the solution.

Consider the following ITCP. It is necessary to find the functions $c(T)$ and $T(x, \tau)$ from the conditions

$$c(T)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right) + K(T)\frac{\partial T}{\partial x} + Q(T),\ x \in (0,\ b),\ \tau \in (0,\ \tau_m); \tag{1}$$

$$T(x,\ 0) = \xi(x),\ x \in (0,\ b); \tag{2}$$

$$T(x,\ t) = F(x),\ t = \text{const} \in (0,\ \tau_m),\ x \in (0,\ b); \tag{3}$$

$$T(b,\ \tau) = f(\tau),\ \tau \in (0,\ \tau_m); \tag{4}$$

$$-\lambda \left.\frac{\partial T}{\partial x}\right|_{x=b} = \varphi(\tau),\ \tau \in (0,\ \tau_m); \tag{5}$$

$$T(0,\ \tau) = g(\tau),\ \tau \in (0,\ \tau_m); \tag{6}$$

$$-\lambda \left.\frac{\partial T}{\partial x}\right|_{x=0} = \psi(\tau)_1\ \tau \in (0,\ \tau_m). \tag{7}$$

Here $\lambda(T)$, $K(T)$, $Q(T)$, $\xi(x)$, $F(x)$, $f(\tau)$, $\varphi(\tau)$, $g(\tau)$, and $\psi(\tau)$ are known functions.

The above statement of the ITCP can have practical applications, for instance in using nondestructive methods for determining or monitoring the thermophysical characteristics of

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materials. We encounter fairly often situations where placement of thermal data units inside the solid under investigation (for instance, a part of a certain structure) is impossible or very difficult, but measurements of the temperature and the thermal flux density at its surface are possible. Additional conditions concerning the temperature distribution in the solid F(x) at the instant of time $t > 0$ involve, of course, certain difficulties, but, in many cases, they can be taken into account in performing experiments. In particular, we shall consider the case where $k = Q = 0$. Assume that the boundary planes of a plate are thermostatically controlled at different temperature levels during the time $0 \le \tau \le t$, while the duration of this interval t is based on the condition that the temperature field has reached steady-state conditions. Then, if during the period $0 \le \tau \le t$ it can be considered that $\lambda(T) = \lambda^* = \text{const}$, a linear temperature distribution F(x) that is known beforehand will be established along the plate thickness.

Remark 1. As follows from the theorem given below, the function $\xi(x)$ can be considered, theoretically, to be unknown. As stated, problem (1)-(7) is assumed to have been assigned for two reasons:

1) Thermophysical experiments can usually be organized so that the initial temperature of the solid T(x, 0) is a known constant;

2) exclusion of one of the functions [$\xi(x)$ in our case] from the number of the quantities to be determined improves considerably the conditionality of the computational form of the inverse problem and markedly increases the accuracy of its solution.

Let us write the uniqueness theorem for problem (1)-(7). We first introduce some notation. For $\tau_m > 0$, $Q_{\tau_m} = (0, b) \times (0, \tau_m)$, and $Q_{t,\tau_m} = (0, b) \times (t, \tau_m)$. For any function $u(x, \tau)$ defined over the set Q_{τ_m} or at its boundary, $\sigma(u)$ is the region of values of the function u, while $\sigma_t(u)$ is the set of values of the function $u(x, \tau)$ when (x, τ) run through the set Q_{t, τ_m}. By $G(\bar{Q}_{\tau_m})$ or, correspondingly, $G(\bar{Q}_{t,\tau_m})$ we denote the set of functions $u(x, \tau)$ having in Q_{τ_m} (or correspondingly in Q_{t, τ_m}) the derivatives u_τ, u_x, u_{xx}, $u_{\tau x}$, $u_{\tau xx}$, and $u_{\tau\tau}$, which are continuous in $\bar{Q}_{\tau_m}$ ($\bar{Q}_{t,\tau_m}$), where the bar over the set symbol denotes closure, i.e., $\bar{Q}_{\tau_m} = [0, b] \times [0, \tau_m]$ and $\bar{Q}_{t,\tau_m} = [0, b] \times [t, \tau_m]$.

THEOREM 1. Assume that, in (1)-(7), $\lambda \ge \lambda_0$, $c \ge c_0$, $\lambda_0 = \text{const} > 0$, $c_0 = \text{const} > 0$, the function $\lambda(z)$ is continuously differentiable, and the functions c(z), K(z), and Q(z) are continuous in the domain of their definition. We also assume that the function $T(x, \tau)$ is strictly monotonic with respect to x and τ in Q_{t,τ_m}:

$$\inf_{Q_{t,\tau_m}} \left|\frac{\partial T}{\partial x}\right| > 0, \tag{8}$$

$$\inf_{Q_{t,\tau_m}} \left|\frac{\partial T}{\partial \tau}\right| > 0. \tag{9}$$

We can then find not more than one vector function $(c, T) \in C(\bar{\sigma}_t(T)) \times G(\bar{Q}_{t,\tau_m})$ satisfying (1)-(7), such that $T \in G(\bar{Q}_{\tau_m})$.

Remark 2. It is understood here that $T \in G(\bar{Q}_{\tau_m})$, but the field is uniquely defined only in Q_{t,τ_m}. If, instead of (8) and (9), we require that $\inf\limits_{Q_{\tau_m}}|\partial T/\partial x| > 0$ and $\inf\limits_{Q_{\tau_m}}|\partial T/\partial \tau| > 0$, the function $T(x, \tau)$ can then be defined in Q_{τ_m}, while c(z) can be defined for $z \in \sigma(T)$, and the function $\xi(x)$ is considered to be unknown in all cases. The domain $\sigma_t(T)$ [$\sigma(T)$ in the second case] is not known beforehand. It is determined in the process of solving the problem.

We shall indicate the basic points of the proof. Without loss of generality, we can assume that, in Q_{t,τ_m},

$$\frac{\partial T}{\partial x} \geqslant \alpha > 0, \ \alpha = \text{const}. \tag{10}$$

We introduce a new independent variable and function by effecting the substitution $(T, x) \leftrightarrow (w, z)$ in accordance with $T(w(z, \tau), \tau) = z$. Using the theorem concerning implicit functions, we obtain, after simple transformations,

$$c(z)\,w_\tau = \frac{\lambda(z)}{w_z^2} w_{zz} + Q(z)\,w_z + \lambda'(z) + K(z), \tag{11}$$

$$w(z,\ t) = p(z),\ \min_{[0,b]}(F(x)) \leqslant z \leqslant \max_{[0,b]}(F(x)), \tag{12}$$

$$w|_{z=f(\tau)} = b, \tag{13}$$

$$-\left.\frac{\partial w}{\partial z}\right|_{z=f(\tau)} = \lambda(f(\tau))\,\frac{1}{\varphi(\tau)}, \tag{14}$$

$$w|_{z=g(\tau)} = 0, \tag{15}$$

$$-\left.\frac{\partial w}{\partial z}\right|_{z=g(\tau)} = \lambda(g(\tau))\,\frac{1}{\psi(\tau)}, \tag{16}$$

$$g(\tau) < z < f(\tau),\ t < \tau < \tau_m. \tag{17}$$

The function p(z) in (12) is known. It constitutes the solution of the functional equation F(p(z)) = z.

In view of the continuity of the $\partial T/\partial x$ function, we consider that (11) is satisfied for $\tau \in [t-\varepsilon, \tau_m]$ and $x \in [0, b]$ for a certain small $\varepsilon > 0$. The relationships (11)-(17) are thus satisfied in the region (with mobile boundaries)

$$g(\tau) < z < f(\tau),\ t-\varepsilon < \tau < \tau_m. \tag{18}$$

Assume that there are two solutions of problem (1)-(7). There are then two solutions of problem (11)-(17): (c_1, w_1) and (c_2, w_2). With regard to the function $v = w_1 - w_2$, we can obtain in domain (18) a linear equation containing $c = c_1 - c_2$ [13]. We then apply the method of Carlemann estimates [13]. The detailed proof is very cumbersome and is not given here. Some details are given in [11].

THEOREM 2. Theorem 1 remains valid also in the case where the function K(T) or Q(T) is sought along with c(T). However, condition (9) in formulating the theorem must be rejected and the function K(Q) substituted for the function c. If we assume that $g(\tau)$ = const is a known number in (6), the function $\psi(\tau)$ in (7) can be considered to be unknown and, instead of (8), it must be stipulated that $\inf\limits_{Q_{t,\tau_m}} \partial T/\partial x > 0$, i.e., that the function $T(x, \tau)$ increase strictly monotonically with respect to x.

The proof of Theorem 2 is provided by using the same method.

Remark 3. Conditions (8) and (9) for the monotony of the function $T(x, \tau)$ can readily be secured under actual conditions. Theoretically, it is necessary to impose additional conditions on the functions appearing in (1)-(7) in order to utilize the maximum principle.

Remark 4. Similar theorems also are valid if one or both boundaries of the region are mobile, i.e., if $\psi_2(\tau) < x < \psi_1(\tau), \tau \in (0, \tau_m), \psi_1$ and $\psi_2 \in C^2[0, \tau_m]$ in (1)-(6).

The inverse problem (1)-(7) is improperly stated, and its solution must be regularized in some way [14, 15].

The regularizing algorithm for the inverse problem under consideration can be developed on the basis of iteration regularization [16]. According to this method, the following iteration algorithm is formed for solving the operator equation Au = f, $u \in U, f \in F$ (U and F are normalized spaces):

$$u^{k+1} = F_A(u^k,\ \Delta f,\ h),\ k = 0,\ 1,\ \ldots,$$

which minimizes the rate of deviation of the left-hand side of the equation from a straight line in the metrics of space F:

$$J(u) = \|A_h u - f_\delta\|_F^2,$$

and the iteration number is used as the regularization parameter (here $\Delta f = f_\delta - f$ is the "noise" in the right-hand side, and h is the parameter of operator approximation). It was found that the gradient methods of steepest descent, minimum discrepancies, and conjugate gradients can be used as the iteration algorithms. The number of the last iteration in these methods is determined with respect to the discrepancy criterion:

$$k^*: J(u_{k^*}) \simeq \delta^2, \ \delta = \|\Delta f\|_F$$

under the assumption that the approximation error is much smaller than the error in assigning the right-hand side.

For the case of a linear operator A and Hilbert spaces U and F, there were obtained in [17] theorems concerning the regularizability of the gradient methods and the stability conditions of the corresponding approximations in using the discrepancy criterion. For nonlinear problems, to which statement (1)-(7) belongs, this approach was substantiated by means of numerical simulation [16, 18-20].

In correspondence with the uniqueness conditions, we shall assume that the function c(T) is continuous and use this assumption as a priori information. This condition is taken into account by choosing a suitable class of functions, among which the required representative is sought, namely, we shall assume that c(T) belongs to the Sobolev space W_2^1 (it is known that $W_2^1 \subset C$).

We shall further assume that the function c(T) is unknown throughout the region $\sigma(T)$ [no special difficulties are encountered in passing to the case where c(T) is known for the T(x, τ) values corresponding to $(x, \tau) \in (0, b) \times (0, t)$, and is unknown only in the $\sigma_t(T)$ region].

The smallest and the largest temperature values for the $\sigma(T)$ range in the case of a homogeneous thermal conductivity equation with the assigned functions $\xi(x)$, $g(\tau)$, and $f(\tau)$ are known beforehand, which follows from the maximum principle. For instance, assume that $\partial T/\partial x < 0$ for all $\tau \in (0, \tau_m)$, and $\partial T/\partial \tau > 0$ for all $x \in (0, b)$. Then, using the inverse problem solution, it is necessary to determine the relationship c(T) in the temperature range (T_0, T_M), where $T_0 = f(0)$, $T_M = g(\tau_m)$.

We introduce the discrepancy functional in the form of the sum of the temperature deviations $T|_0 = T(c(T), 0, \tau)$, $T|_b = T(c(T), b, \tau)$, and $T|_t = T(c(T), x, t)$, from the functions $g(\tau)$, $f(\tau)$, and $F(x)$, respectively, in the metrics of space L_2:

$$J(c) = \int_0^{\tau_m} [T|_0 - g(\tau)]^2 d\tau + \int_0^{\tau_m} [T|_b - f(\tau)]^2 d\tau + \int_0^b [T|_t - F(x)]^2 dx.$$

Here $T(c(T), x, \tau)$ is the solution of the boundary-value problem of thermal conductivity in the region Q_{τ_m} for the assigned functions $\xi(x)$, $\varphi(\tau)$, and $\psi(\tau)$ and a certain function c(T).

Following the method of iteration regularization and assuming that the function $\varphi(\tau)$ is known exactly, we shall consider the extremum problem of determining c(T) within the framework of the (1)-(7) model on the basis of the condition

$$\min [J(c) - \delta_J^2],$$

where δ^2_J is the allowable level of discrepancy minimization, which is determined by the errors in assigning $g(\tau)$, $f(\tau)$, and F(x).

We shall seek the solution c(T) in parametrized form:

$$\tilde{c}(T) = \sum_{j=1}^n c_j \eta_j(T),$$

where $\{\eta_j(T)\}_1^n$ is a system of basis functions, and $\{c_j\}^n_1$ are the sought coefficients.

Cubic V-splines can be used as $\eta_j(T)$. The functions of this class form a subspace in the space W^ν_2 ($\nu \leq 3$) and are convenient for solving ITCPs [18, 19].

Thus, the extremum problem is related to the determination of the vector $\bar{c} \in R^n$ on the basis of the conditions

$$\min_{\bar{c}} [\hat{J}(\bar{c}) - \delta_J^2], \ \hat{J}(\bar{c}) = \int_0^{\tau_m} [\hat{T}|_0 - g(\tau)]^2 d\tau + \int_0^{\tau_m} [\hat{T}|_b - f(\tau)]^2 d\tau +$$

$$+\int_0^b [\hat{T}|_t - F(x)]^2 dx, \quad \hat{T}|_l = T(\bar{c}(T), l, \tau), \ \hat{T}|_t = T(\bar{c}(T), x, t).$$

The iteration determination of the sought quantities is performed according to the algorithm

$$\bar{c}^{k+1} = \bar{c}^k - \alpha_k \bar{p}^k, \ k = 0, 1, \ldots, k^*, \tag{19}$$

where $k^* : J(\bar{c}_{k*}) \simeq \delta^2_J$.

The direction of descent $\bar{p}$ must be chosen so as to secure the convergence of approximations at the number of space W_2^1 [21]. The coefficients α_k in expression (19) are calculated at each iteration on the basis of the condition

$$\alpha_k : \min_\alpha \hat{J}(\bar{c}^k - \alpha \bar{p}^k).$$

In obtaining an estimate of the sought function c(T), we calculate the corresponding temperature field $T(x, \tau)$.

Remark 5. We have considered above the overdetermined statement of the ITCP. We shall now assume that there is no information on $\xi(x)$. In this case, considering F(x) as the initial condition and assigning the target functional in the form $J(c) = \int_0^{\tau_m} [T|_0 - g(\tau)]^2 d\tau + \int_0^{\tau_m} [T|_b - f(\tau)]^2 d\tau$, we can find, on the basis of the above method, a pair of functions, c(T) and $T(x, \tau)$, with the definition domains $\sigma_t(T)$ and Q_{t,τ_m}, respectively.

The described approach to writing an algorithm for solving coefficient ITCPs can also be generalized to include other gradient methods satisfying the condition of regularizability with respect to the number of iterations as well as inverse problems of determining the vector functions $\{K(T), T(x, \tau)\}$ and $\{Q(T), T(x, \tau)\}$.

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AXISYMMETRIC BENDING OF A HEATED CIRCULAR PLATE ON AN ELASTIC BASE WITH ACCOUNT OF ITS DEFORMABILITY OVER ITS THICKNESS

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The axisymmetric problem is solved for the bending of a circular plate on a heated half-space under the action of a distributed load and a temperature field.

One of the founders of the theory of bending of beams and plates on an elastic base is Proktor [1, 2], who, in 1919, formulated a computational process for the reduction of the problem of bending of a narrow beam on a half-space to the solution of an integrodifferential equation taking account of the elastic deformations of contiguous bodies. Because the series of solutions that he obtained proved to be weakly convergent [2], another variant of this method was formulated, based on integral account of the crumpling of a beam over its thickness [3, 4]. As was shown by these calculations, taking account of the crumpling of a beam over thickness leads to a considerable redistribution of the reaction pressure under the base of the beam. Below, we give a further development of Proktor's method applied to circular plates resting on an elastic half-space.

1. We consider a circular plate of radius a, on the bounding planes of which the external loads and temperature are constant:

$$\begin{aligned}\sigma_z &= -q,\ T = T_1 = \mathrm{const}\ \text{ for }\ z = -h,\\ \sigma_z &= -p,\ T = T_2 = \mathrm{const}\ \text{ for }\ z = h.\end{aligned} \tag{1}$$

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